

# Resonant Wave-Particle Interaction in Nonuniform Plasmas

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The appearance of resonant-particle effects in the linear dynamics of electron waves in a plasma whose electrons are trapped in a potential trough is discussed and conditions for the characteristic times involved are derived. For the special case of a parabolic potential trough it is shown that, within a limited time interval, a strong resonant effect may occur, due to the interaction of a wave with all particles, which is not contained in the solutions given so far.

The conditions for the appearance of resonant-particle effects in the linear dynamics of plasma waves, leading to Landau damping and micro-instability, have been studied in detail for homogeneous systems<sup>1-4</sup>. Much less attention has been paid, however, to the case of nonuniform plasmas, where in general a numerical approach has been adopted<sup>5-7</sup>. The appearance of poles in the expression for the perturbed particle distribution function has been noted by HARKER<sup>6</sup>, the physical conditions under which the associated resonant-particle effects occur have, however, not been discussed.

It is the scope of this paper to display these conditions for a simple, but typical case, using an approach which has been previously applied for investigating the limiting case of waves in a homogeneous plasma bounded by electrodes<sup>8</sup>. We consider longitudinal electron waves in a one-dimensional plasma immersed in a nonuniform static electric field  $E_0(x)$ , associated with a potential  $\Phi(x)$  which falls off monotonically on either side of a point, say,  $x = 0$  (see Fig. 1).

The linearized Vlasov equation

$$\frac{\partial f_1}{\partial t} + v \frac{\partial f_1}{\partial x} - \frac{e}{m} E_0 \frac{\partial f_1}{\partial v} - \frac{e}{m} E_1 \frac{\partial f_0}{\partial v} = 0 \quad (1)$$

where  $f_0$  and  $f_1$  are, respectively, the unperturbed and perturbed electron distribution function,  $E_1$  the electric field perturbation, and  $e$  and  $m$  the electron

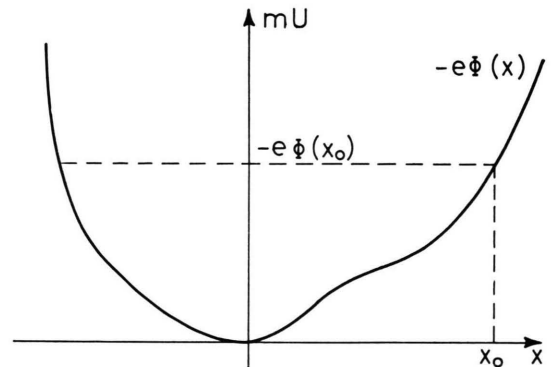


Fig. 1. Electron of energy  $-e\Phi(x_0)$ , trapped within the potential  $\Phi(x)$ .

charge and mass, can be solved in the case at hand by introducing, instead of the position  $x$  and the velocity  $v$  of the particles,

$$U = \frac{v^2}{2} - \frac{e}{m} \Phi(x),$$

and

$$\tau = \int^x dx/v$$

as independent variables<sup>6,9</sup>. We explicitly consider a single mode of the form

$$E_1(x, t) = A(x) e^{-i\omega_0 t}. \quad (3)$$

For simplicity, we assume furthermore that in a limited range of electron energies  $mU$  (to be identified later with the “resonant” domain) the initial

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$$\text{condition} \quad f_1(U, \tau, t=0) = 0, \quad (4)$$

holds<sup>10</sup>. By this assumption only effects are eliminated that depend on the detailed structure of the initial perturbation in velocity space and that, as a consequence, are macroscopically irreproducible.

Noting that  $f_0 = f_0(U)$  and that  $f_1(U, \tau, t)$  is

$$\begin{aligned} f_1(U, \tau, t) = & \frac{e}{m} \frac{df_0(U)}{dU} \left\{ \int_0^\tau A(U, \tau') v(U, \tau') \exp\{-i\omega_0(t - \tau + \tau')\} d\tau' \right. \\ & - \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{i T [\omega_0 - (2k\pi/T)]} \int_0^{\frac{T}{2}} \left\{ A(U, \tau') v(U, \tau') \left[ 1 - \exp i \left( \omega_0 - \frac{2k\pi}{T} \right) \left( t - \tau + \tau' - \frac{T}{2} \right) \right] \cdot e^{-i\omega_0(t - \tau + \tau' - T/2)} \right. \\ & \left. \left. + A(U, -\tau') v(U, -\tau') \left[ 1 - \exp i \left( \omega_0 - \frac{2k\pi}{T} \right) \left( t - \tau - \tau' + \frac{T}{2} \right) \right] \cdot e^{-i\omega_0(t - \tau - \tau' + T/2)} \right\} d\tau' \right\}. \end{aligned} \quad (6)$$

The dynamics of the modes, that is the value of  $\omega_0$  is determined by relations containing integrals of the form

$$I = \int dU f_1(U, \tau, t) g(U). \quad (7)$$

If one uses, e. g., the Maxwell equation

$$\partial E_1 / \partial t = 4\pi e \int v f_1 dv, \quad (8)$$

one needs  $I$  with  $g = 1$ . Usually (see, e. g., Ref. <sup>9</sup>), in these integrals only the terms proportional to  $e^{-i\omega_0 t}$ , are retained, replacing

$$\left[ 1 - \exp i \left( \omega_0 - \frac{2k\pi}{T} \right) \left( t - \tau \pm \tau' \mp \frac{T}{2} \right) \right] \text{ by } 1.$$

In fact, deforming the contour of the integration

periodic in  $\tau$  with a period

$$T = T(U) \equiv \oint (dx/v) \quad (5)$$

where the integration runs over a closed particle orbit, one obtains, integrating by a standard method and expanding singular functions of  $\omega_0$  in principal parts connected with the singularities,

over  $U$  into a Landau-like path, defined by

$$\text{Im}[1/T(U)] < 0,$$

the integral over "free-streaming" terms, proportional to  $\exp\{-i(2k\pi/T)t\}$  tends to 0 for large times. Then the second term of Eq. (6) has poles for  $U = U_0$  such that

$$\frac{2k\pi}{T(U_0)} = \omega_0, \quad k \neq 0, \quad (9)$$

describing resonant wave-particle interactions. Their effect can be associated with the contribution of the correspondig pole to the integral  $I$ , which, for  $|\text{Im} \omega_0|$  small, is readily calculated to be

$$\begin{aligned} I_k = & \frac{(-1)^k T(U_0)}{|2k(dT/dU)_{U=U_0}|} \frac{e}{m} \left( \frac{df_0(U)}{dU} \right)_{U=U_0} g(U_0) \int_0^{\frac{T}{2}} \left\{ A(U_0, \tau') v(U_0, \tau') \exp\{-i\omega_0(t - \tau + \tau' - T(U_0)/2)\} \right. \\ & \left. + A(U_0, -\tau') v(U_0, -\tau') \exp\{-i\omega_0(t - \tau - \tau' + T(U_0)/2)\} \right\} d\tau'. \end{aligned} \quad (10)$$

The physical content of this result can be better seen, however, using the complete expression (6). It is important to notice that the second term of this expression, as written above, has *no poles*. For  $|\text{Im} \omega_0(t - \tau \pm \tau' \mp T/2)| \ll 1$ , it has, however, a resonant behaviour, if

$$\left| \left( \beta_0 - \frac{2k\pi}{T(U)} \right) \left( t - \tau \pm \tau' \mp \frac{T(U)}{2} \right) \right| < \pi \quad (11)$$

where  $\beta_0 \equiv \text{Re} \omega_0$ . Relation (11) contains the definition of the energy interval of electrons which interact resonantly with the wave of frequency  $\omega_0$ . Since  $\tau$  and  $\tau'$  have maximum values of the order of  $T(U)$ , relation (11) implies essentially

$$|\beta_0 - 2k\pi/T(U)| t < \pi, \quad (11')$$

if  $t \gg T$ . The resonant interval shrinks, hence, with increasing  $t$  (cf. <sup>8</sup>). For  $t$  large enough that  $T(U)$  can be replaced by its linear approximation around  $U = U_0$ , one obtains for the width of the resonant interval explicitly

$$\Delta U = \frac{T^2(U_0)}{|2k(dT/dU)_{U=U_0}|} \cdot t^{-1}. \quad (12)$$

<sup>10</sup> The general solution of Eq. (1) with  $E_1$  considered given and  $f_1(U, \tau, t=0)$  nonvanishing is obtained from the solution deduced here by adding a term which describes the evolution of the initial distribution by free streaming of the particles and formally satisfies Eq. (1) with  $E_1 \equiv 0$ ; it depends, hence, only on the initial condition, but not on  $E_1$ .

Supposing  $\Delta U \ll \bar{U}$  (13)

where  $\bar{U}$  is the typical scale of variation in  $U$  of  $f_1(U, \tau, t) g(U)$  ( $\omega_0 - 2k\pi/T$ ), i. e., in simple cases, a quantity of the order of the thermal energy, and expanding

$$\exp\{i(\omega_0 - 2k\pi/T)(t - \tau \pm \tau' \mp T/2)\}$$

to first order in the exponent, the contribution of the range (11) to the integral I, connected with the  $k$ -th term of the sum in Eq. (6), can readily be estimated. The result coincides with Eq. (10). This shows that resonant particle effects are due to particles within the interval (12) and appear for

$$t \gg T(U_0), \quad T^2(U_0)/|2kU(dT/dU)_{U=U_0}|. \quad (14)$$

The latter condition implies that  $t$  must be larger than the period of the motion of the resonant particles and such that the number of resonant particles has become small compared to the total particle number.

A special case appears when the potential  $\Phi(x)$  is *parabolic*, because then the period  $T$  of the particle motion is independent of  $U$  so that Eqs. (10) and (12), as well as the second part of Eq. (14) make no longer sense. From the usual treatment one would infer (cf., e. g., <sup>6</sup>) that resonant effects cannot occur as long as

$$\beta_0 \neq 2k\pi/T,$$

because then no poles are present in the integrand of  $I$ . On the other hand, for  $\beta_0 = 2k\pi/T$  a "super-resonance" of the wave with all particles would appear. This latter case has no practical meaning, however, because there is no eigenvalue of this kind <sup>5, 6</sup>.

Hence, it has been concluded that resonant effects do not exist in the case of a parabolic potential  $\Phi(x)$ .

The above treatment, using the complete expression (6), shows, however, that strong resonant effects may occur also for  $\beta_0 \neq 2k\pi/T$ . Conditions (11) and (11'), respectively, are now to be considered as defining an upper limit for the time during which the strong resonance of all particles with the wave persists. Their appearance, for given  $\beta_0$ , is, hence, subject to the condition

$$T \ll t < \frac{\pi}{|\beta_0 - 2k\pi/T|}. \quad (15)$$

During this time,  $I_k$  turns out to be proportional to  $t$ , showing that the "super-resonance" induces a damping or growth which is faster than exponential. Only for times larger than those allowed for by relation (15) the wave dynamics tends, on the average, to the one described by the solutions obtained by WEIBEL <sup>5</sup> and HARKER <sup>6</sup>. It is to be noted, however, that for these times the wave dynamics, in general, will be subject to a beating effect between the wave frequency  $\beta_0$  and the  $k$ -th harmonic of the particle frequency  $2\pi/T$  for which

$$|\beta_0 T - 2k\pi| \ll \pi$$

is satisfied, as evident from Eq. (6). The strength of this beating effect is dependant on the choice of the initial condition  $f_1(U, \tau, t=0)$ . The case explicitly considered by Weibel and Harker is the one where the initial condition is chosen such that the beating effect vanishes. For long times, however, also nonlinear effects tend to be important.